AP Calculus

Integration by U-Substitution

Name________________________

Integration Day 9

1-4: Evaluate the indefinite integral.

1. $\int \cot x \, dx$

$u = \sin x$

$du = \cos x \, dx$

$\int \frac{\cos x}{\sin x} dx$

$\int \frac{1}{u} du = \ln|u| + C = \boxed{\ln|\sin x| + C}$

2. $\int x^2 \sqrt{2+x} \, dx$

$u = 2 + x$ $\quad x = u - 2$

$du = dx$

$\int (u-2)^2 \sqrt{u} \, du$

$= \int (u^2 - 4u + 4) u^{1/2} du$

$= \int u^{5/2} - 4u^{3/2} + 4u^{1/2} du$

$= \frac{2}{7} u^{7/2} - \frac{8}{5} u^{5/2} + \frac{8}{3} u^{3/2} + C$

$= \boxed{\frac{2}{7}(2+x)^{7/2} - \frac{8}{5}(2+x)^{5/2} + \frac{8}{3}(2+x)^{3/2} + C}$

3. $\int x(2x+5)^8 \, dx$

$u = 2x + 5$ $\quad x = \frac{u-5}{2}$

$du = 2dx$

$\frac{du}{2} = dx$

$\int \frac{u-5}{2} \cdot u^8 \cdot \frac{du}{2}$

$= \frac{1}{4} \int (u-5) \cdot u^8 \, du$

$= \frac{1}{4} \int (u^9 - 5u^8) du$

$= \frac{1}{4}\left(\frac{u^{10}}{10} - \frac{5u^9}{9} + C\right)$

$= \frac{u^{10}}{40} - \frac{5u^9}{36} + C$

$= \boxed{\frac{(2x+5)^{10}}{40} - \frac{5(2x+5)^9}{36} + C}$

4. $\int x^3 \sqrt{x^2+1} \, dx$

$u = x^2 + 1$ $\quad x^2 = u - 1$

$du = 2x \, dx$

$\frac{du}{2} = x \, dx$

$\int (u-1)\sqrt{u} \frac{du}{2}$

$= \frac{1}{2} \int (u-1) u^{1/2} du$

$= \frac{1}{2} \int (u^{3/2} - u^{1/2}) du$

$= \frac{1}{2}\left(\frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} + C\right)$

$= \boxed{\frac{1}{5}(x^2+1)^{5/2} - \frac{1}{3}(x^2+1)^{3/2} + C}$

5-16: Evaluate the definite integral

5. $\int_0^1 \cos\left(\frac{\pi t}{2}\right) dt$

$u = \frac{\pi t}{2}$

$du = \frac{\pi}{2} dt$

$\frac{2}{\pi} du = dt$

$\int_{u=0}^{u=\pi/2} \cos u \cdot \frac{2}{\pi} du$

$\frac{2}{\pi} \int_0^{\pi/2} \cos u \, du = \frac{2}{\pi} \sin u \Big|_0^{\pi/2}$

$= \frac{2}{\pi} \sin\left(\frac{\pi}{2}\right) - \frac{2}{\pi} \sin(0) = \boxed{\frac{2}{\pi}}$

6. $\int_0^1 (3t-1)^{50} dt$

$u = 3t - 1$

$du = 3dt$

$\frac{du}{3} = dt$

$\int_{u=-1}^{u=2} u^{50} \cdot \frac{du}{3}$

$= \frac{1}{3} \int_{-1}^{2} u^{50} du = \frac{1}{3} \cdot \frac{u^{51}}{51} \Big|_{-1}^{2}$

$= \frac{u^{51}}{153} \Big|_{-1}^{2}$

$= \boxed{\frac{2^{51}}{153} + \frac{1}{153}}$

7. $\int_0^1 \sqrt[3]{1+7x} \, dx$

$u = 1 + 7x$

$du = 7dx$

$\frac{du}{7} = dx$

$\int_{u=1}^{u=8} \sqrt[3]{u} \cdot \frac{du}{7}$

$= \frac{1}{7} \int_1^8 u^{1/3} du = \frac{1}{7} \cdot \frac{3}{4} u^{4/3} \Big|_1^8$

$8^{4/3} = (\sqrt[3]{8})^4$

$= \frac{3}{28}(8^{4/3} - 1^{4/3})$

$= \frac{3}{28}(16 - 1) = \frac{3}{28}(15)$

$= \boxed{\frac{45}{28}}$

8. $\int_0^{\pi} \sec^2\left(\frac{t}{4}\right) dt$

$u = \frac{t}{4}$

$du = \frac{1}{4} dt$

$4du = dt$

$\int_{u=0}^{u=\pi/4} \sec^2 u \cdot 4du$

$= 4 \int_0^{\pi/4} \sec^2 u \, du$

$= 4 \tan u \Big|_0^{\pi/4} = 4\tan\left(\frac{\pi}{4}\right) - 4\tan(0)$

$= 4 - 0 = \boxed{4}$

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9. $\int_1^2 \frac{e^{\frac{1}{x}}}{x^2}\,dx$

$u=\frac{1}{x}=x^{-1}$

$du=-x^{-2}\,dx$

$du=-\frac{1}{x^2}\,dx$

$-du=\frac{1}{x^2}\,dx$

$\int_{u=1}^{u=1/2} e^u \cdot -du$

$=-\int_1^{1/2} e^u\,du=-e^u\Big|_1^{1/2}=\boxed{-e^{1/2}+e}$

10. $\int_0^1 xe^{-x^2}\,dx$

$u=-x^2$

$du=-2x\,dx$

$\frac{du}{-2}=x\,dx$

$\int_{u=0}^{u=-1} e^u \cdot \frac{du}{-2}$

$=-\frac{1}{2}\int_0^{-1} e^u\,du=-\frac{1}{2}e^u\Big|_0^{-1}$

$=-\frac{1}{2}e^{-1}+\frac{1}{2}=\boxed{\frac{-1}{2e}+\frac{1}{2}}$

11. $\int_0^{\frac{\pi}{2}} \cos x\sin(\sin x)\,dx$

$u=\sin x$

$du=\cos x\,dx$

$\int_{u=0}^{u=1} \sin u\,du=-\cos u\Big|_0^1$

$=-\cos(1)+\cos 0$

$=\boxed{-\cos(1)+1}$

12. $\int_0^a x\sqrt{a^2-x^2}\,dx$

$u=a^2-x^2$

$du=-2x\,dx$

$\frac{du}{-2}=x\,dx$

$\int_{u=a^2}^{u=0} \sqrt{u}\cdot\frac{du}{-2}$

$=-\frac{1}{2}\int_{a^2}^{0} u^{1/2}\,du=\frac{-1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_{a^2}^{0}$

$=-\frac{1}{3}(0)^{3/2}+\frac{1}{3}(a^2)^{3/2}=\boxed{\frac{1}{3}a^3}$

13. $\int_0^a x\sqrt{x^2+a^2}\,dx$

$u=x^2+a^2$

$du=2x\,dx$

$\frac{du}{2}=x\,dx$

$\int_{u=a^2}^{u=2a^2} \sqrt{u}\cdot\frac{du}{2}$

$=\frac{1}{2}\int_{a^2}^{2a^2} u^{1/2}\,du=\frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_{a^2}^{2a^2}$

$=\frac{1}{3}(2a^2)^{3/2}-\frac{1}{3}(a^2)^{3/2}=\frac{2\sqrt{2}}{3}a^3-\frac{1}{3}a^3$

$=\boxed{\frac{2\sqrt{2}-1}{3}a^3}$

14. $\int_1^2 x\sqrt{x-1}\,dx$

$u=x-1$ $\quad x=u+1$

$du=dx$

$\int_{u=0}^{u=1} (u+1)\sqrt{u}\,du$

$=\int_0^1 (u+1)u^{1/2}\,du=\int_0^1 (u^{3/2}+u^{1/2})\,du$

$=\frac{2}{5}u^{5/2}+\frac{2}{3}u^{3/2}\Big|_0^1$

$=\frac{2}{5}+\frac{2}{3}-(0)=\frac{6}{15}+\frac{10}{15}=\boxed{\frac{16}{15}}$

15. $\int_{-3}^{0} \frac{2x}{(x^2+1)^2}\,dx$

$u=x^2+1$

$du=2x\,dx$

$\int_{u=10}^{u=1} \frac{1}{u^2}\,du$

$=\int_{10}^{1} u^{-2}\,du=\frac{u^{-1}}{-1}\Big|_{10}^{1}=\frac{-1}{u}\Big|_{10}^{1}$

$=-1+\frac{1}{10}=\boxed{\frac{-9}{10}}$

16. $\int_0^1 18x^2(3x^3-4)^2\,dx$

$u=3x^3-4$

$du=9x^2\,dx$

$2du=18x^2\,dx$

$\int_{u=-4}^{u=-1} u^2\cdot 2\,du$

$=2\int_{-4}^{-1} u^2\,du=\frac{2u^3}{3}\Big|_{-4}^{-1}$

$=\frac{2(-1)^3}{3}-\frac{2(-4)^3}{3}$

$=-\frac{2}{3}+\frac{128}{3}=\boxed{\frac{126}{3}}$